Exercise set 3 – Kinematics – Solutions

Reminders

Simplified notation of sines and cosines

To simplify the notation, we use:

- $\sin(\theta) = s$
- $\cos(\theta) = c$
- $\sin(\theta_1) = s_1$
- $\cos(\theta_1) = c_1$
- $\sin(\theta_2) = s_2$
- $\cos(\theta_2) = c_2$
- $\cos(\theta_1 + \theta_2) = c_{1+2}$
- $\sin(\theta_1 + \theta_2) = s_{1+2}$

Rotation and translation matrices

Recall that:

- $\mathbf{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ describes the rotation of θ around the origin (in 2D)
- $\mathbf{R}_x(\theta)$ describes the rotation of θ around the axis x
- $\mathbf{R}_y(\theta)$ describes the rotation of θ around the axis y
- $\mathbf{R}_z(\theta)$ describes the rotation of θ around the axis z
- $\mathbf{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$ describes the translation vector $\mathbf{t}$

Sequence of transformations

The sequence $a \rightarrow b \rightarrow c$ describes the transformation a followed by the transformation b followed by the transformation c.

Quaternions

The quaternion $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} \lambda_0 \\ \lambda_x \\ \lambda_y \\ \lambda_z \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \boldsymbol{\lambda} \end{pmatrix} = \lambda_0 + i\lambda_x + j\lambda_y + k\lambda_z$$

describes a rotation with a rotation axis $\boldsymbol{\lambda}$ and a rotation angle θ such that $\lambda_0 = \cos(\theta/2)$ and $\boldsymbol{\lambda} = \sin(\theta/2)[x,y,z]^T$ with $||[x,y,z]||=1$.

Scalar product:

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|cos(\vec{u},\vec{v})$

Cross product:

Let $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$. Then $\vec{u} \times \vec{v} = \begin{pmatrix} u_y v_z - v_y u_z \\ v_x u_z - u_x v_z \\ u_x v_y - v_x u_y \end{pmatrix}$

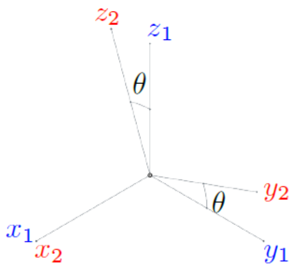
Exercise 1

Find the matrices of pure rotation around the three axes of the Cartesian system:

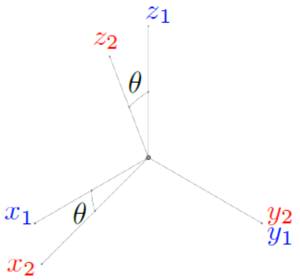
1. $\mathbf{R_x}$ around *x*.
2. $\mathbf{R_y}$ around *y*.
3. $\mathbf{R_z}$ around *z*.

Solution 1

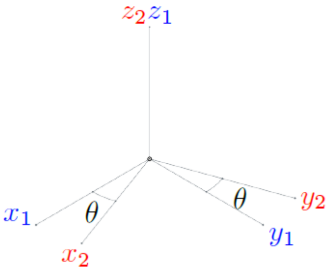
We start by drawing the rotations in Cartesian frames; in blue - the initial coordinate system, and in red - the final coordinate system, in order to write the generic rotation matrix:



(a) rotation around *x*



(b) rotation around *y*



(c) rotation around *z*

The generic 3D rotation matrix is written as:

$$\boldsymbol{R} = \begin{pmatrix} \overrightarrow{x_2}.\overrightarrow{x_1} & \overrightarrow{y_2}.\overrightarrow{x_1} & \overrightarrow{z_2}.\overrightarrow{x_1} \\ \overrightarrow{x_2}.\overrightarrow{y_1} & \overrightarrow{y_2}.\overrightarrow{y_1} & \overrightarrow{z_2}.\overrightarrow{y_1} \\ \overrightarrow{x_2}.\overrightarrow{z_1} & \overrightarrow{y_2}.\overrightarrow{z_1} & \overrightarrow{z_2}.\overrightarrow{z_1} \end{pmatrix}$$

By calculating each scalar product, we obtain:

1. $\mathbf{R_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$
2. $\mathbf{R_y} = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$
3. $\mathbf{R_z} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise 2

Consider the following two sequences of operations:

$$\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$$

$$\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$$

Give the rotation matrices corresponding to these sequences. Are they equivalent?

Solution 2

We calculate the rotation matrices for the two sequences:

$$\mathbf{R_1} = \mathbf{R_y}(90^\circ)\mathbf{R_z}(90^\circ) = \begin{pmatrix} c_{90^\circ} & 0 & s_{90^\circ} \\ 0 & 1 & 0 \\ -s_{90^\circ} & 0 & c_{90^\circ} \end{pmatrix} \begin{pmatrix} c_{90^\circ} & -s_{90^\circ} & 0 \\ s_{90^\circ} & c_{90^\circ} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{R_2} = \mathbf{R_z}(90^\circ)\mathbf{R_y}(90^\circ) = \begin{pmatrix} c_{90^\circ} & -s_{90^\circ} & 0 \\ s_{90^\circ} & c_{90^\circ} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{90^\circ} & 0 & s_{90^\circ} \\ 0 & 1 & 0 \\ -s_{90^\circ} & 0 & c_{90^\circ} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

Therefore, these two sequences are not equivalent!

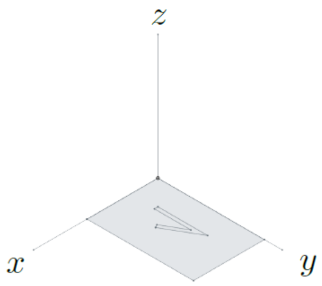
Exercise 3

Consider the two sequences from the previous exercise:

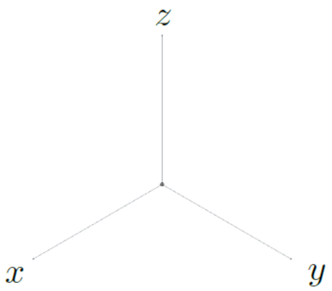
$$\mathbf{R_z}(90^\circ) \rightarrow \mathbf{R_y}(90^\circ)$$

$$\mathbf{R_y}(90^\circ) \rightarrow \mathbf{R_z}(90^\circ)$$

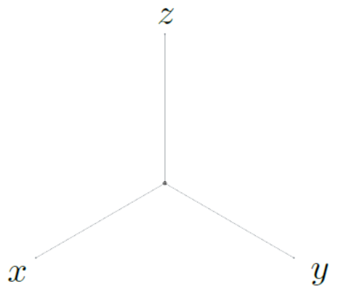
For each of the two sequences, determine graphically by iteration the result of the sequence using an object oriented in a Cartesian coordinate system in isometric projection, as in the figure below (‘b’ and ‘c’ are to be completed by you):



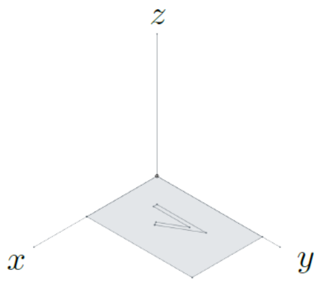
(a) object in initial position



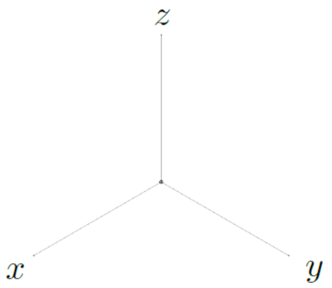
(b) object after $\mathbf{R_z}(90^\circ)$, to be completed



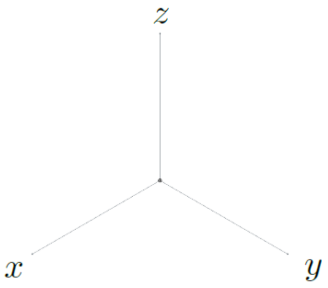
(c) object after $\mathbf{R_z}(90^\circ) \rightarrow \mathbf{R_y}(90^\circ)$, to be completed



(a) object in initial position



(b) object after $\mathbf{R_y}(90^\circ)$, to be completed

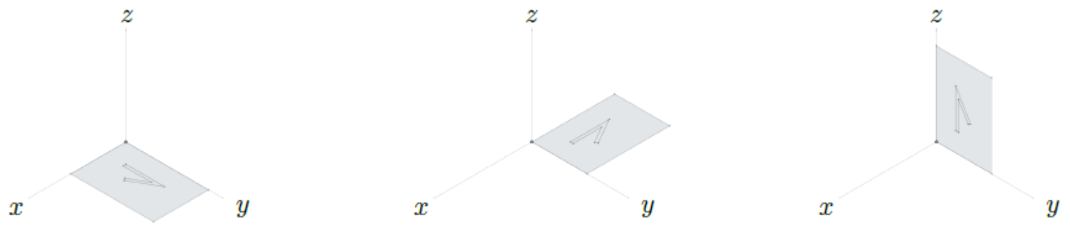


(c) object after $\mathbf{R_y}(90^\circ) \rightarrow \mathbf{R_z}(90^\circ)$, to be completed

Solution 3

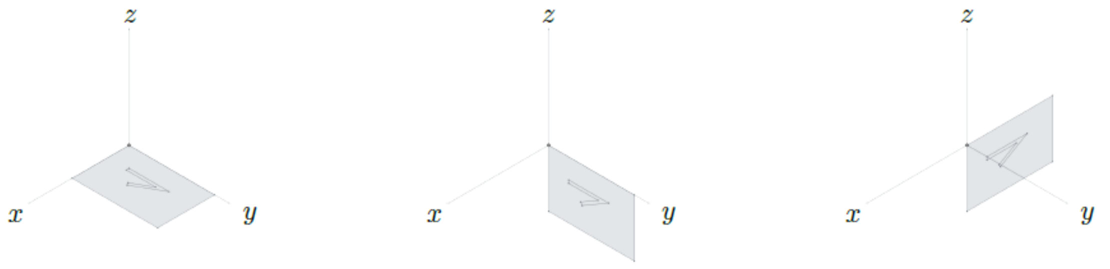
Be careful, rotations are not linked to the body frame!

By iteration, we find:



(a) object in initial position (b) object after $\mathbf{R}_z(90^\circ)$ (c) object after $\mathbf{R}_z(90^\circ) \rightarrow \mathbf{R}_y(90^\circ)$

Figure 2 - First sequence : $\mathbf{R}_z(90^\circ) \rightarrow \mathbf{R}_y(90^\circ)$.

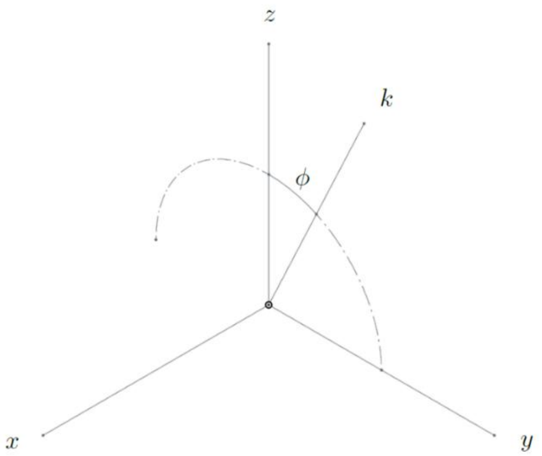


(a) object in initial position (b) object after $\mathbf{R}_y(90^\circ)$ (c) object after $\mathbf{R}_y(90^\circ) \rightarrow \mathbf{R}_z(90^\circ)$

Figure 3 - Second sequence: $\mathbf{R}_y(90^\circ) \rightarrow \mathbf{R}_z(90^\circ)$.

Exercise 4

Find the matrix of direction cosines for a rotation with an angle θ around an axis k, which is in the yz plane and which is inclined by an angle ϕ with respect to the axis z, i.e find the rotation matrix corresponding to this transformation. **Hint:** use a sequence of basic rotations ($\mathbf{R_x}$, $\mathbf{R_y}$ or $\mathbf{R_z}$).



Solution 4

To find the solution, we perform a first rotation of angle ϕ around the axis x in order to align the axis k with the axis z, then we rotate by θ around the axis z, and finally we rotate by an angle -ϕ around the axis x to "return" to the correct frame of reference:

$$\begin{aligned}
\mathbf{R} &= \mathbf{R_x}(-\phi)\mathbf{R_z}(\theta)\mathbf{R_x}(\phi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{-\phi} & -s_{-\phi} \\ 0 & s_{-\phi} & c_{-\phi} \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{pmatrix} \\
&= \begin{pmatrix} c_\theta & -s_\theta c_\phi & s_\theta s_\phi \\ s_\theta c_\phi & c^2_\phi c_\theta + s^2_\phi & c_\phi s_\phi (1 - c_\theta) \\ -s_\theta s_\phi & c_\phi s_\phi (1 - c_\theta) & s^2_\phi c_\theta + c^2_\theta \end{pmatrix}
\end{aligned}$$

Exercise 5

Consider the two sequences of exercises 1 and 2:

$$\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$$

$$\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$$

For each of these sequences:

1. Determine the resulting corresponding quaternion.
2. Deduce:
   - (a) the corresponding angles of rotation.
   - (b) the corresponding unit axes of rotation.

Solution 5

1. We start by calculating $\mathbf{Q}_{\mathbf{y}90^\circ}$ and $\mathbf{Q}_{\mathbf{z}90^\circ}$, the quaternions corresponding respectively to $\mathbf{R}_{\mathbf{y}}(90^\circ)$ and $\mathbf{R}_{\mathbf{z}}(90^\circ)$:

   For $\mathbf{Q}_{\mathbf{y}90^\circ}$, we have:

   - $\theta_y = 90° \rightarrow \cos(\theta_y/2) = \sin(\theta_y/2) = \frac{\sqrt{2}}{2}$
   - $\boldsymbol{\lambda_y} = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}$ ($\boldsymbol{\lambda}_y$ is the axis of rotation whose norm is $\sin(\theta_y/2)$)
   - $\lambda_{y0} = \cos(\theta_y/2) = \frac{\sqrt{2}}{2}$

   And finally:

$$\mathbf{Q}_{\mathbf{y}90^\circ} = \begin{pmatrix}\lambda_{y0}\\ \boldsymbol{\lambda_y}\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

   For $\mathbf{Q}_{\mathbf{z}90^\circ}$, we have:

   - $\theta_z = 90° \rightarrow \cos(\theta_z/2) = \sin(\theta_z/2) = \frac{\sqrt{2}}{2}$
   - $\boldsymbol{\lambda_z} = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$ ($\boldsymbol{\lambda}_z$ is the axis of rotation whose norm is $\sin(\theta_z/2)$)
   - $\lambda_{z0} = \cos(\theta_z/2) = \frac{\sqrt{2}}{2}$

   And finally:

$$\mathbf{Q}_{\mathbf{z}90^\circ} = \begin{pmatrix}\lambda_{z0}\\ \boldsymbol{\lambda_z}\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

   We notice that the two quaternions are unitary (the opposite would have been surprising).

We then calculate the two sequences by multiplying the quaternions (product which is of course non-commutative):

First sequence: $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$

$$\mathbf{Q_1} = \mathbf{Q}_{\mathbf{y}90°}\mathbf{Q}_{\mathbf{z}90}$$

$$= \begin{pmatrix} \lambda_{y0}\lambda_{z0} - \boldsymbol{\lambda_y} \cdot \boldsymbol{\lambda_z} \\ \lambda_{y0}\boldsymbol{\lambda_z} + \lambda_{z0}\boldsymbol{\lambda_y} + \boldsymbol{\lambda_y} \times \boldsymbol{\lambda_z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} \\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_{1,0}\\ \boldsymbol{\lambda_1}\end{pmatrix}$$

Second sequence : $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$\mathbf{Q_2} = \mathbf{Q}_{\mathbf{z}90°}\mathbf{Q}_{\mathbf{y}90°}$$

$$= \begin{pmatrix} \lambda_{z0}\lambda_{y0} - \boldsymbol{\lambda_z} \cdot \boldsymbol{\lambda_y} \\ \lambda_{z0}\boldsymbol{\lambda_y} + \lambda_{y0}\boldsymbol{\lambda_z} + \boldsymbol{\lambda_z} \times \boldsymbol{\lambda_y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_{2,0}\\ \boldsymbol{\lambda_2}\end{pmatrix}$$

The two resulting quaternions are unitary as expected.

2. (a) The calculation of the angles is given below:

First sequence: $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$

$$\theta_1 = 2\arccos\left(\lambda_{1,0}\right) = 2\arccos\left(1/2\right) \text{ et } \theta_1 = 2\arcsin\left(||\boldsymbol{\lambda_1}||\right) = 2\arcsin\left(\sqrt{3}/2\right)$$

$$\Rightarrow \theta_1 = \frac{2\pi}{3} rad = 120°$$

Second sequence: $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$\theta_2 = 2\arccos\left(\lambda_{2,0}\right) = 2\arccos\left(1/2\right) \text{ et } \theta_2 = 2\arcsin\left(||\boldsymbol{\lambda_2}||\right) = 2\arcsin\left(\sqrt{3}/2\right)$$

$$\Rightarrow \theta_2 = \frac{2\pi}{3} rad = 120°$$

(c) Obtaining the (unitary) axes is as follows:

First sequence: $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$

$$\begin{aligned}\mathbf{k_1} &= \frac{\boldsymbol{\lambda_1}}{\sin(\theta_1/2)} \\ &= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\end{aligned}$$

Second sequence: $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$

$$\begin{aligned}\mathbf{k_2} &= \frac{\boldsymbol{\lambda_2}}{\sin(\theta_2/2)} \\ &= \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}\end{aligned}$$

Exercise 6

Consider an object with vertices A, B, C, transformed in such a way that its vertices are found at A', B', C'; the vectors giving the coordinates of the points are **a**, **b**, **c**, **a’**, **b’** and **c’**:

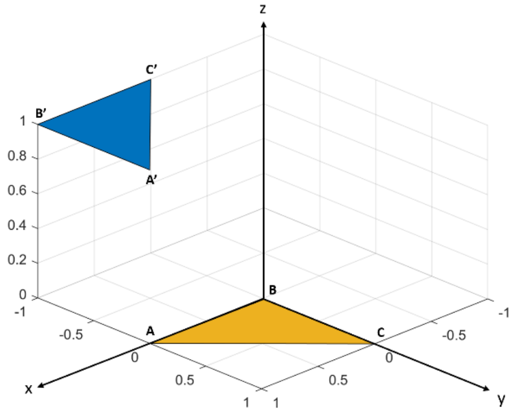
$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}\ \mathbf{b} = \begin{pmatrix}0\\0\\0\end{pmatrix}\ \mathbf{c} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$\mathbf{a}' = \begin{pmatrix}1\\0\\1\end{pmatrix}\ \mathbf{b}' = \begin{pmatrix}1\\-1\\1\end{pmatrix}\ \mathbf{c}' = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

1. Find the rotation (angle and axis) and the translation (offset and axis) corresponding to the transformation. **Hint:** use a drawing.
2. Deduce the corresponding homogeneous transformation matrix.

Solution 6

Note: There could be several possible rotations and translations to obtain the final configuration. We will choose one of the axes of rotation and corresponding translation to that one. For example, we could rotate around the axis z and then translate it to the final position, or we can rotate around a vertical axis passing through point A and use the corresponding translation.

1. In this solution we will proceed with the second aforementioned example.



By making a drawing, we notice that A, B and C are on a horizontal plane ($z = 0$), just like A', B' and C' ($z = 1$). It is therefore necessary to rotate around a vertical axis, then translate by 1 along the axis z.
The points A and A' have the same coordinates x and y, therefore we deduce that the axis of rotation passes through A. The angle of rotation is found to be 90°.
Finally, there is a translation $\mathbf{t}$ of 1 along the axis z, and a rotation $\mathbf{R}$ of 90°about a vertical axis passing through A.

2. It is possible to find the homogenous matrix by calculating the matrices corresponding to transformations identified by the drawing. For this, we first calculate the matrix $\mathbf{R}$ of rotation around the vertical axis passing through A and the matrix $\mathbf{t}$ of translation of 1 along the axis z.

   Then, we build the homogeneous matrices of transformations associated with this rotation and with this translation, $\mathbf{M_{z,A,90°}}$ and $\mathbf{M_{z,1}}$ respectively; finally, we build the homogeneous matrix $\mathbf{M}$ of the transformation:

$$\mathbf{R} = \mathbf{R_z}(90°) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{t_x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{t_x} - \mathbf{R}\mathbf{t_x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

The homogeneous matrix of the rotation is therefore:

$$\mathbf{M_r} = \begin{pmatrix} \mathbf{R} & \mathbf{t_x} - \mathbf{R}\mathbf{t_x} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The homogeneous matrix of the translation is therefore:

$$\mathbf{M_t} = \begin{pmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, the homogeneous matrix of the transformation is:

$$\mathrm{M} = \mathrm{M_t}\mathrm{M_r} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can now verify this transformation, using the homogeneous coordinates (we add an extension factor of 1 at the end of the Cartesian coordinate vector) and the homogeneous transformation matrix:

$$\begin{pmatrix}\mathbf{a}'\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0 & -1 & 0 & 1\\1 & 0 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \mathbf{M}\begin{pmatrix}\mathbf{a}\\1\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{b}'\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}0 & -1 & 0 & 1\\1 & 0 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \mathbf{M}\begin{pmatrix}\mathbf{b}\\1\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{c}'\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}0 & -1 & 0 & 1\\1 & 0 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \mathbf{M}\begin{pmatrix}\mathbf{c}\\1\end{pmatrix}$$

The homogeneous transformation matrix $\mathbf{M}$ corresponds well to the described transformation.